

IMPARTIAL COLORING GAMES

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ABSTRACT. Coloring games are combinatorial games where the players alternate painting uncolored vertices of a graph one of $k > 0$ colors. Each different ruleset specifies that game's coloring constraints. This paper investigates six impartial rulesets (five new), derived from previously-studied graph coloring schemes, including proper map coloring, oriented coloring, 2-distance coloring, weak coloring, and sequential coloring. For each, we study the outcome classes for special cases and general computational complexity. In some cases we pay special attention to the Grundy function.

1. INTRODUCTION AND BACKGROUND

This paper deals with impartial coloring games on graphs. Roughly speaking, two players alternate coloring the vertices of a graph (under some specific constraints), until no more vertices can be painted. Following the normal play convention, the first player unable to color a vertex loses the game [2]. In each of these rulesets, a description of the game state consists of a graph, a number of colors and a partial coloring of the vertices, adhering to the coloring restrictions of the ruleset. The options for each position are those where exactly one new vertex has been colored.

Definition 1. An *impartial coloring ruleset* is an impartial combinatorial game ruleset where players alternate coloring the vertices of a graph. The state of the game consists of: a graph, $G = (V, E)$, which may be either directed or undirected; a positive integer, k , representing the number of colors; and a partial coloring of the vertices, $c : V \rightarrow \{1, \dots, k\} \cup \{\text{'uncolored'}\}$. Impartial coloring rulesets include some sort of restriction to which c must adhere. The options for each position are those which have the same state aside from a new coloring, c' where $V = V' \cup \{v_0\}$, $c'(V') = c(V')$, $c'(v_0) \neq c(v_0)$ and $c(v_0) = \text{'uncolored'}$.

We define many such rulesets, analyze outcome classes for different situations, and study the computational complexity of each. In section 2, we study the rules for PROPER k -COLORING, based on proper map-colorings. In section 3, we use oriented colorings on digraphs [8] to define the games ORIENTED k -COLORING and ORIENTED BLUE-RED-COLORING. In section 4, we base WEAK k -COLORING on

weak colorings, where each non-isolated vertex must be adjacent to a vertex of at least one other color [5]. In section, 5, we use the 2-distance coloring defined in [9] to define the ruleset 2-DISTANCE COLORING. The last game, SEQUENTIAL COLORING, defined first in [3], is covered in section 6, including an improvement to the original solution algorithm on paths and cycles.

The rest of this section is devoted to basic combinatorial game theory definitions, with a focus on impartial game theory. Those who already have an understanding of the topic (known as “gamesters”) may safely skip to the next section.

1.1. Combinatorial Game Theory Basics. These standard definitions can be found in [2]. The set of nonnegative integers will be denoted by \mathbb{N} . For all $U \subset \mathbb{N}$, $\text{Mex}(U)$ denotes the smallest non-negative integer not included in U . We will denote by \mathbb{G} the set of positions of a given game. If $u \in \mathbb{G}$, then $O(u)$ denotes the *options* of u , i.e., the set of positions reachable by a legal move from u .

In combinatorial game theory, *partisan* games are those for which both players do not always have the same options from a position. The investigation of partisan coloring games on graphs led to the map-coloring game (introduced by M. Gardner in 1981), and the corresponding game chromatic number [1, 7]. One can also refer to the game of *Col*, which is a partisan coloring game on planar graphs [2].

Alternatively, impartial games are those for which both players always have the same options. Notably, the impartial game Nim was completely solved [4] in 1905. In the case of impartial coloring games, there are few results in the literature. In the 90’s, Bodlaender investigated online (or sequential) coloring games [3, 12, 16]. One can also consider NODE-KAYLES [11] as an impartial coloring game with a unique color. In [3], it is shown that finding a winning strategy for some coloring games, such as the sequential coloring game, or NODE-KAYLES [11] are PSPACE-complete.

Given a short impartial combinatorial game (i.e., an impartial game where any position cannot be visited twice), positions are classified as \mathcal{P} or \mathcal{N} : it is a \mathcal{P} -position if the second player has a winning strategy, and an \mathcal{N} -position otherwise (the first player has a winning strategy). In addition, Sprague and Grundy introduced a function $\mathcal{G} : \mathbb{G} \rightarrow \mathbb{N}$ called the Grundy function, such that $\forall u, v \in \mathbb{G}$

- $\mathcal{G}(u) = 0$ if and only if u is a \mathcal{P} -position,
- $\mathcal{G}(u) > 0$ if and only if u is a \mathcal{N} -position,
- $\mathcal{G}(u) = \mathcal{G}(v)$ if and only if playing on the sum $u + v$ is a \mathcal{P} -position.

According to Sprague and Grundy [15], it turns out that $\forall u, \mathcal{G}(u) = \text{Mex}(\mathcal{G}(O(u)))$.

In terms of complexity, our objective is also to have a better idea of what can be expected for each of these games. Therefore, we will mainly focus our research on simple families of graphs such as paths and cycles. As much as possible, we determine the computational complexity of these games, evidence of the difficulty for players to choose optimal moves on their turn. (See [6] for more information on computational complexity and its relevance to combinatorial game theory.).

2. PROPER K-COLORINGS

Definition 2. PROPER k -COLORING is an impartial coloring ruleset where no two adjacent vertices of the graph may be painted the same color.

When $k = 1$, the game is known as NODE-KAYLES [2], which is known to be PSPACE-complete [14]. As we show next, the general game is also PSPACE-complete for all k .

Proposition 1. *Determining whether an instance of PROPER k -COLORING is an \mathcal{N} -position is PSPACE-complete.*

Proof. Since all instances of the game are short and the number of plays is bounded above by the number of unpainted vertices, the problem is in PSPACE. To show hardness, we will reduce from NODE-KAYLES. The reduction is straightforward; we simply connect each original Kayles vertex to new vertices, each painted one of the $k - 1$ colors.

Given an instance of NODE-KAYLES, $G = (V, E)$, create a new graph, $G' = (V', E')$ where

$$V' = \{v_0, \dots, v_{k-1} : \forall v \in V\}, \text{ and}$$

$$E' = \{(v_0, v_i) : \forall v \in V, i \in \{1, \dots, k-1\}\} \cup \{(u_0, w_0) : \forall (u, w) \in E\}.$$

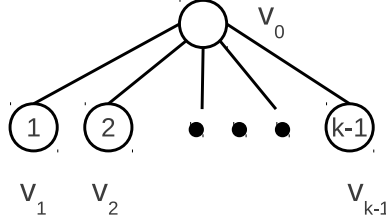


FIGURE 1. Reduction of vertex, v , from NODE-KAYLES to the PROPER k -COLORING GADGET. v_0 is uncolored.

To complete the reduction, we paint all vertices v_i where $i > 0$ with the i^{th} color. Now, only the vertices v_0 remain unpainted, all with only a single color available. These are adjacent to neighbors just as their counterparts, v are. Thus, each move in this game is equivalent to the corresponding move on the NODE-KAYLES instance. So, this game is also PSPACE-hard. Note that this reduction preserves the planarity of the original graph.

Together, determining whether a position of PROPER k -COLORING is in outcome class \mathcal{N} is PSPACE-complete. \square

The following proposition provides a strong symmetry argument to solve the PROPER k -COLORING GAME on several families of graphs.

Proposition 2. *Let $G = (V, E)$ be a graph. If there exists an involution, s , of G satisfying:*

- $\exists! u \in V$ such that $s(u) = u$
- $\forall v \in V, (v, s(v)) \notin E$,

then the starting position of PROPER k -COLORING on G is in \mathcal{N} .

Proof. Let us play the first move on the unique vertex u satisfying $s(u) = u$. We will recursively show that the first player is able to hold the following property (P1) during the game:

(P1) : For all vertex $v \in V \setminus \{u\}$, the vertices v and $s(v)$ hold the same color before second player's turn.

The property (P1) is true after the first move on u . Assume (P1) is true after any odd number of moves. Without loss of generality, suppose that the second player paints some vertex $v_0 \in V$ with the color $i \in \{1 \dots k\}$. Since (P1) was true before this move, then $s(v_0)$ is not colored yet. Hence the first player chooses to color $s(v_0)$ with the color i . This is allowed since s is an involution: indeed, suppose there exists some vertex $v_2 = s(v_1)$, adjacent to $s(v_0)$, and colored with i . Since (P1) was true before the second player's move, v_1 is also colored with i . But since $v_2 = s(v_1)$ is adjacent to $s(v_0)$, the vertex v_1 is adjacent to v_0 , which yields a contradiction with the proper coloring constraint. Consequently we can color $s(v_0)$ with i , and then (P1) holds.

Thus, the first player always has a legal move after the second player's turn. \square

For $k = 2$, Proposition 2 has a similar result for deciding whether a game position is an \mathcal{N} -position.

Proposition 3. *Let $G=(V,E)$ be a graph. If there exists an involution, s , without a fixed point, the starting position of PROPER 2-COLORING with colors $\{R, B\}$ is in \mathcal{P} .*

Proof. We will use a similar proof to Proposition 2, by showing that the second player is able to hold the following property (P2) during the game :

(P2) : After each even move, for each colored vertex v , the image, $s(v)$, is painted the opposite color.

Trivially, (P2) is true before the first move. Inductively, assume (P2) is true after any even number of moves. Now suppose, without loss of generality, that the first player paints some vertex v_0 with the color B . Hence we can color $s(v_0)$ with R : suppose indeed there exists some vertex $v_2 = s(v_1)$, adjacent to $s(v_0)$, and colored with R . As (P2) was true before the first player's move, v_1 is colored with B , and since s is an involution, the vertices v_1 and v_0 are adjacent, and coloring v_0 with B was not a legal move. Therefore we can color $s(v_0)$ with R . \square

These two propositions allow us to come to further conclusions about the status of the game positions on several families of graphs. For example, when playing on paths with two colors, we have the following result:

Corollary 4. *PROPER 2-COLORING starting positions are \mathcal{N} positions when played on paths of odd length, and \mathcal{P} positions on paths of even length.*

Proof. In the case of paths of odd length, let us label the vertices from 1 to $2k - 1$ for any positive integer k . Let s be a function on the vertex, such that $s(i) = 2k - i$. It is easy to verify that s is an involution having a unique fixed point, with no edge between any vertex u and $s(u)$. Hence, according to Proposition 2, the position is a first-player win.

In the case of paths of even length with vertices labeled from 1 to $2k$, we consider the function s defined on V such that $s(i) = 2k - i + 1$. It is easy to verify that s

is an involution without a fixed point. Hence, according to Theorem 3, the game is a loss for the first player. \square

Corollary 5. *PROPER 2-COLORING starting positions are in \mathcal{P} when played on cycles.*

Proof. For cycles of even length, the proof is similar as in Corollary 4. In the case of odd cycles, Propositions 2 and 3 cannot be applied. We thus prove differently that there is a winning strategy for the second player: Let v_0 be the first vertex colored (say with the color B) by the first player. Without loss of generality, let v_1 be an adjacent vertex of v_0 and v_2 the other adjacent vertex of v_1 . The second player then colors v_2 with R . Hence, v_1 cannot be colored any more. As v_1 can not be colored, we can delete it from the graph. The second player can now apply the strategy given in the proof of Proposition 2 as if they played on a path of even length with both extremities having opposite colors. \square

By the same methods, Propositions 2 and 3 solve the proper coloring game for other classes of graphs. Given any positive integer k , the PROPER k -COLORING starting positions are in \mathcal{N} in the following cases:

- grids of size $n \times m$, when $n \times m$ is odd
- k -dimensional grids, when all dimensions are odd
- complete binary trees
- k -complete trees when k is odd
- odd-length paths

By Proposition 3, the following classes of graphs yield PROPER 2-COLORING starting positions in \mathcal{P} :

- hypercubes
- grids of size $n \times m$, when $n \times m$ is even
- k -dimensional grids, with at least one even dimension
- paths and cycles of even length

In addition, the Grundy function is fully determined in the case of paths and cycles. The only non-trivial case is for paths of odd length, which is detailed below.

Proposition 6. *For any starting position, G , of PROPER COLORING on a path of odd length, $\mathcal{G}(G) = 1$.*

Proof. Let us consider a coloring game ($G1$) on a path of odd length $2k + 1$, added to a coloring game ($G2$) on a path of length 1. We label the vertices of the path on ($G1$) from 1 to $2k + 1$. The game sum $G1 + G2$ is a losing game. Consider the following nearly-symmetric strategy for the second player:

- If the first player plays on node $k + 1$, then color the node of $G2$
- If the first player plays on $G2$, color the node $k + 1$
- If the first player plays on node i , play on $2(k + 1) - i$ with the same color

Now we must ensure that the second player always has a move. It is clear that in the first case, player 2 can play. In the second case, node $k + 1$ is not colored yet (because $G2$ was not yet colored). We can color node $k + 1$ because if any adjacent node i is colored (k or $k + 2$), the node $2(k + 1) - i$ is colored in the same color, and $k + 1$ can be colored. In the last case, the coloration of the two sides of the paths (from 1 to k and $2k + 1$ to $k + 2$) are similar due to strategy property, and

then if the first players can play, the second player can maintain the similarity and play too. \square

Open Problems. Despite these results, there are simple families of graphs which remain hard to solve, such as caterpillars (i.e., trees in which all the vertices are within distance 1 of a central path), or k -complete trees when k is even. Moreover, deciding whether there exists an involution in a graph is a hard problem in general.

3. ORIENTED COLORINGS

In the rest of the paper, our main idea is to consider variants of PROPER K -COLORING by changing the coloring rules according to existing coloring graph parameters. Our study will generally be focused on paths and cycles. The first variant we investigate is a game based on oriented colorings, as described in [8].

Definition 3. ORIENTED k -COLORING is an impartial coloring ruleset where for each arc (u, v) in digraph G , the vertices u and v are not painted with the same color, and there is no other arc, (v', u') of E where u is painted the same as u' and v the same as v' .

The oriented coloring of graphs has never been considered in the context of combinatorial games. One of the difficulties of ORIENTED k -COLORING is that the orientation of the colors is decided during the game (more precisely, when two adjacent vertices are colored for the first time). Hence we also choose to consider the variant of this game with $k = 2$ and where the orientation is fixed before the first move:

Definition 4. ORIENTED BLUE-RED-COLORING is an impartial coloring ruleset where $k = 2$ (we will refer to the colors as Blue and Red) and for each arc (u, v) of digraph, G : if both u and v are colored, then u must be Blue, and v must be Red.

This choice considerably simplifies the game on paths and cycles since coloring a vertex v in Blue (resp. Red) forbids playing on the predecessor (resp. successor) of v . Hence coloring a vertex from an uncolored path P is an operation that cuts P into two shorter paths, with their extremities possibly colored.



FIGURE 2. Example of oriented path of length 5

Thus, it is easy to remark that any game position of ORIENTED BLUE-RED-COLORING on a path is a disjunctive sum of paths belonging to the four categories below:

- The A class : paths with the first vertex, v_1 , colored Blue. Denote by A_k a path of A of length k .
- The B class : paths with the last vertex, v_k , colored Red. Denote by B_k a path of B of length k .
- The C class : paths with the first vertex, v_1 , colored Blue, and the last, v_k , painted Red. Denote by C_k a path of C of length k .
- The D class : paths with no vertex colored. Denote by D_k a path of D of length k .

We first give equivalences between the A and B classes, followed by a similar result concerning C and D classes. In both cases, we let the vertices of a path of length k be labeled from v_1 to $v_k : v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$.

Proposition 7. *For all integers $k > 0 : \mathcal{G}(A_k) = \mathcal{G}(B_k)$.*

Proof. Recursively, we prove that $\mathcal{G}(A_k) = \mathcal{G}(B_k)$ by showing that for any option of A_k , there exists an option of B_k with the same Grundy value (note that every option on B_k has a equivalent option in A_k by a similar proof).

Base Case: It is clear that $\mathcal{G}(A_1) = \mathcal{G}(B_1) = 0$; there are no moves available.

Suppose that for each $1 \leq i < k$, we have $\mathcal{G}(A_i) = \mathcal{G}(B_i)$. We will now show that $\mathcal{G}(A_k) = \mathcal{G}(B_k)$.

An option of A_k is given by coloring the vertex v_i ($2 \leq i \leq k$) in red or blue. We prove that coloring v_i is equivalent to coloring v_{k-i+1} on B_k with the opposite color:

- If v_i is colored Blue, then A_k will be splitted into two shorter paths A_{i-2} and $A_{k-(i-1)}$. Painting v_{k-i+1} Red on B_k splits into B_{k-i+1} and B_{i-2} . Hence we conclude the hypothesis.
- If v_i is colored Red, then A_k will be splitted into two shorter paths C_i and $D_{k-(i+2)}$. Painting v_{k-i+1} Blue on B_k splits into D_{k-i-2} and C_i .

□

Proposition 8. *For all integers $k > 0 : \mathcal{G}(C_{k+3}) = \mathcal{G}(D_k)$.*

Proof. We use the same technique as in the proof of Proposition 7.

Base Case ($k = 1$): $\mathcal{G}(C_4) = \mathcal{G}(D_1) = 1$, because all options of either game have no further moves.

By induction, we show that for any option of C_{k+3} , there exists an option of D_k with the same Grundy value: playing Blue (resp. Red) on v_i from C_{k+3} is equivalent to playing Blue (resp. Red) on v_{i-2} (resp. v_{k-i+2}) from D_k . Reciprocally, playing Blue (resp. Red) on v_i from D_k is equivalent to playing Blue (resp. Red) on v_{i+2} (resp. v_{k-i+2}) from C_{k+3} . □

In order to compute the Grundy function for the ORIENTED BLUE-RED-COLORING GAME on uncolored paths (i.e., paths of the D class), we remark that any move from a position in A, B, C or D leads to a sum of two paths also belonging to A, B, C or D . This leads to the following recursive characterization:

Proposition 9.

$$\begin{aligned} \mathcal{G}(D_k) &= \text{Mex}\{ \\ (1) \quad &\{\mathcal{G}(D_{i-2}) \oplus \mathcal{G}(A_{k+1-i}), \forall 1 \leq i \leq k\} \cup \\ (2) \quad &\{\mathcal{G}(B_i) \oplus \mathcal{G}(D_{k-(i+1)}), \forall 1 \leq i \leq k\} \} \end{aligned}$$

$$\begin{aligned} \mathcal{G}(A_k) &= \text{Mex}\{ \\ (1) \quad &\{\mathcal{G}(A_{i-2}) \oplus \mathcal{G}(A_{k+1-i}), \forall 3 \leq i \leq k\} \cup \\ (2) \quad &\{\mathcal{G}(C_i) \oplus \mathcal{G}(D_{k-(i+1)}), \forall 2 \leq i \leq k\} \} \end{aligned}$$

$$\begin{aligned} \mathcal{G}(C_k) &= Mex\{ \\ (1) \quad &\{\mathcal{G}(A_{i-2}) \oplus \mathcal{G}(C_{k+1-i}), \forall 3 \leq i \leq k-1\} \cup \\ (2) \quad &\{\mathcal{G}(C_i) \oplus \mathcal{G}(B_{k-(i+1)}), \forall 2 \leq i \leq k-2\} \} \end{aligned}$$

Proof. Visually. Example is given in Figure 3. In each Mex formula, the first part (1) corresponds to coloring v_i with Blue, and the second part (2) corresponds to coloring v_i with Red. \square

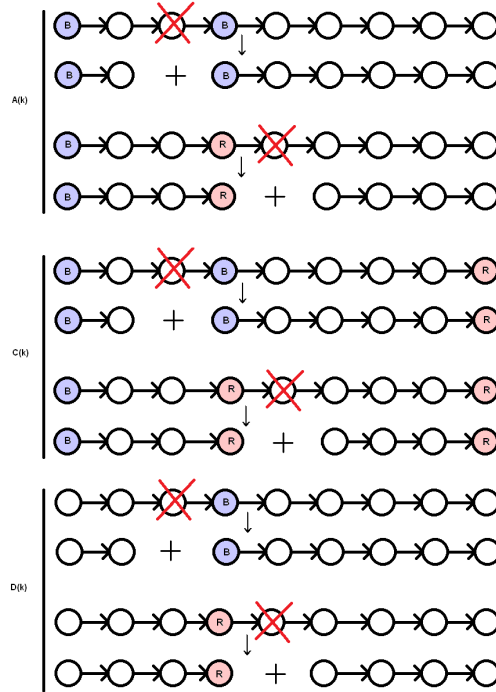


FIGURE 3. Proof of Proposition 9

Unfortunately, the previous proposition does not provide a polynomial time algorithm to compute the value of $\mathcal{G}(D_k)$. Even the zeros of the Grundy function seem hard to characterize. By computing the first values of \mathcal{G} , we note that they split into rare and common values. This phenomenon, which is described in [2], mainly appears for octal games and is frequently used to provide a quick computation of their nimbers. It appears in our study since the recursive characterization described in Proposition 9 has the same structure as those of octal games.

The interest of such a partition of the nimbers is that the rare values form a closed space under nim-addition. The sum of two common values makes a rare value, while adding a rare and a common value gives a common value. In our situation, we observed that the sparse space is the closure (under the Nim-sum operator) of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 24, 40, 64, 136, 264, 520, 1032\}$. By considering this property and together with Propositions 7 and 8, we considerably improved the speed of the algorithm described in Proposition 9. In concrete terms, we computed the Grundy

function of positions in A, B, C and D for paths of lengths 1 up to 10000000. Unfortunately, no periodicity appears in the Grundy function. However, we note the following facts:

- $\forall k > 3$, we have $\mathcal{G}(A_k) > 0$ and $\mathcal{G}(B_k) > 0$.
- We know 26 \mathcal{P} positions for D_k , the last one being at $k = 8084$.
- The maximal found value for $\mathcal{G}(D_k)$ is $\mathcal{G}(3099736) = 1401$.
- The largest known index for a rare value is $k = 642196$.
- There exist odd-length paths which are losing positions for the first player (\mathcal{P}).
- $\mathcal{G}(D_k)$ is not periodic for $k < 10\,000\,000$.

Although the Grundy function is not fully computed for the classes A and B , we can prove that games played on these classes always are \mathcal{N} positions:

Proposition 10. $\forall k > 3 : \mathcal{G}(A_k) > 0$ and $\mathcal{G}(B_k) > 0$.

Proof. Whatever the length of the path, it suffices to prove that there exists an option with Grundy value 0.

Assume k is odd. Then play Blue on $v_{(k+3)/2}$. According to Proposition 9, the Grundy value of the resulting position is equal to $\mathcal{G}(A_{(k+3)/2-2}) \oplus \mathcal{G}(A_{k+1-(k+3)/2}) = \mathcal{G}(A_{(k-1)/2}) \oplus \mathcal{G}(A_{(k-1)/2}) = 0$.

Assume k is even. Then play Blue on $v_{k/2+1}$. The Grundy value is thus $\mathcal{G}(C_{k/2+1}) \oplus \mathcal{G}(D_{k-(k/2+1+1)}) = \mathcal{G}(C_{k/2+1}) \oplus \mathcal{G}(D_{k/2-2})$. Since we know from Proposition 8 that $\mathcal{G}(C_{k/2+1}) = \mathcal{G}(D_{k/2-2})$, the Grundy value of this option is 0. \square

Corollary 11. *Given an integer $l > 3$, ORIENTED BLUE-RED-COLORING starting positions on a cycle of length l is in \mathcal{P} .*

Proof. When first player plays Blue (resp. Red) on a node of the cycle, the predecessor (resp. successor) of the chain can not be painted. Then the chain is equivalent to an A or B case, which is always in \mathcal{N} . Therefore, the second player can win in all cases. \square

Open Problems. There are lot of questions that remain open for this kind of coloring:

- Is $k = 8084$ the last value for which D_k is a \mathcal{P} -position?
- Is there any periodicity of the Grundy function, with period $> 10\,000\,000$?
- Can we provide a polynomial time algorithm to compute the Grundy function?
- What if we consider that the orientation is not fixed before starting the game?

The computational complexity of this game mimics that of PROPER k -COLORING. The proof is a corollary to Proposition 1.

Corollary 12. *Determining whether a position of ORIENTED k -COLORING is in \mathcal{N} is PSPACE-complete.*

This reduction is exactly the same as in Proposition 1 except that all edges are oriented.

Proof. Again, we reduce from NODE-KAYLES. Given a game of NODE-KAYLES with undirected graph $G = (V, E)$, we construct a new directed graph, $G' = (V', E')$ as in the proof of Proposition 1 except that E' consists of *arcs*. (v_0 has arcs leading to

each v_i in that reduction; each other edge can be directed in either direction.) In the game of ORIENTED k -COLORING on G' , only one of the k colors can be used by the players (since all vertices are adjacent to vertices of all other colors) and thus no two adjacent vertices can be colored, just as in the original NODE-KAYLES game. Thus the two games are equivalent and ORIENTED k -COLORING is PSPACE-hard. Since the game has a maximum number of moves, $|V'|$, the game is in PSPACE; together, deciding the outcome class is a PSPACE-complete problem. \square

As it turns out, ORIENTED BLUE-RED COLORING is also a PSPACE-complete game.

Proposition 13. *Determining whether a position of ORIENTED BLUE-RED COLORING is in \mathcal{N} is PSPACE-complete.*

This proof is based off the observation that if there are two vertices connected by arcs in both directions, only one of the pair can be colored.

Proof. As with other proofs, we will reduce from NODE-KAYLES. Given a game of NODE-KAYLES with undirected graph, $G = (V, E)$, we construct a new *directed* graph, $G' = (V', E')$ where:

$$\begin{aligned} V' &= V \text{ and} \\ E' &= (u, v), (v, u) : (u, v) \in E. \end{aligned}$$

Since each pair of adjacent vertices is connected by arcs in both directions in G' , if a vertex is painted either Red or Blue, the adjacent vertices may not be colored at all. Thus, each move on G' is equivalent to the corresponding move on G (ignoring the color). Thus, the position of ORIENTED BLUE-RED COLORING on G' is equivalent to the NODE-KAYLES game on G and the game is PSPACE-hard. Since the maximum number of moves is $|V'|$, the solution can be computed in PSPACE and deciding the outcome class is a PSPACE-complete problem. \square

4. WEAK COLORINGS

We now consider a coloring game on a graph G with the constraint that each game position is a partial weak coloring of G . A weak coloring of a graph G is a vertex coloring such that each non-isolated vertex is adjacent to at least one vertex with different color [5].

Definition 5. WEAK 2-COLORING is an impartial coloring ruleset where two adjacent vertices can be colored with the same color if and only if each vertex is also adjacent to a vertex of the opposite color.

In the field of chromatic graph theory, weak coloring is completely solved, in particular since each graph is 2-weakly colorable.

We here study WEAK 2-COLORING on paths and cycles. In the case of paths and cycles of even length, we can use a generalization of Propositions 2 and 3 to conclude that these instances of WEAK 2-COLORING are all \mathcal{P} positions, while odd paths are \mathcal{N} positions.

Alternative consideration must be applied in the case of odd cycles, which differs from the results for PROPER 2-COLORING.

Theorem 14. WEAK 2-COLORING *starting positions on odd cycles are in \mathcal{N} .*

Proof. For a positive integer k , we consider a cycle of length $2k + 1 = n$. By way of contradiction, suppose there is a winning strategy for the second player. As both players play alternately, there must be at least one vertex, say v , which is not colored at the end of game. Denote by l_0 and r_0 the two adjacent vertices of v , and by l_1, r_1 the other adjacent vertices to l_0 and r_0 respectively. By the same way, we denote by $l_2 \dots l_k$ (resp. $r_2 \dots r_k$) the other adjacent vertices of $l_1 \dots l_{k-1}$ (resp. $r_1 \dots r_{k-1}$). We first remark that at the end of the game, l_0 and r_0 hold necessarily different colors. If not, v could have been colored, contradicting the hypothesis. If l_1 (resp. r_1) has the opposite color of l_0 (resp. r_0), we are able to color v in the same color as l_0 (resp. r_0) contradicting the hypothesis. In addition, since v is not colored, l_0 and r_0 can not be adjacent to a vertex of the same color. Therefore, l_1 and r_1 are uncolored at the end. Moreover, we have $l_1 \neq r_1$ since the cycle has an odd length. By repeating these steps, we show the fact (F) : the vertices l_{2i+1} and r_{2i+1} for $0 \leq i < k/2$ are not colored, while the vertices l_{2i} and r_{2i} for $1 \leq i < k/2$ have opposite colors. We now consider the final adjacent vertices l_k and r_k :

- If k is odd, there are two uncolored adjacent vertices l_k and r_k at the end of the game, which is not possible.
- If k is even, there are two colored adjacent vertices l_k and r_k with different colors. Hence l_{k-1} and r_{k-1} can be colored, contradicting the fact (F).

Consequently, there is no winning strategy for the second player. \square

Corollary 15. *WEAK 2-COLORING starting positions on odd cycles have Grundy value g where g is the parity of the number of uncolored vertices. Thus, the strategies of the players does not matter in the outcome of the game.*

Proof. According to the proof of Theorem 14, each vertex is colored at the end of the game, independent of the moves of both players. \square

Hence, WEAK 2-COLORING is completely solved on paths and cycles. We did not investigate this game on other kinds of graphs, but we think that playing on any graph may not be as hard as the previous problems. This consideration is due to the fact that every graph can be weakly colored with two colors.

Open Problems.

- What is the computational complexity of WEAK 2-COLORING on any graph? Is it polynomial?
- Does the problem become straightforward if there is more than two available colors?

5. 2-DISTANCE COLORINGS

In this section we study a kind of graph coloring called *2-distance coloring*. It has been previously studied in [9]. As before, we study it in the context of impartial combinatorial games.

Definition 6. *d -DISTANCE k -COLORING GAME* is an impartial coloring ruleset where two vertices at distance less than or equal to d cannot be painted the same color.

Our results focus mainly on the case where $d = 2$.

Theorem 16. 2-DISTANCE 2-COLORING *starting positions are in \mathcal{P} when G is an even length path or an even length cycle.*

Proof. Playing 2-DISTANCE 2-COLORING on a graph $G = (V, E)$ is equivalent to playing PROPER 2-COLORING on $G' = (V, E')$ where $E' = E \cup \{(u, v) : d(u, v) = 2\}$. The notation $d(u, v)$ corresponds to the (edge)-distance between two vertices. When G is a path or a cycle, by applying the same proof as for Proposition 3 on G' , we can conclude it is in \mathcal{P} . Indeed, if we label the vertex of G' from 1 to $2k$, the involution that we consider is $s(i) = 2k - i + 1$. \square

The case of odd length paths and cycles remains open in the general case. We implemented a recursive algorithm to compute the outcome class (\mathcal{P} or \mathcal{N}) of odd paths for lengths between 3 and 17. We did not manage to compute it for higher lengths, because of the complexity of the algorithm. Here are the first results:

Length	Status
3	\mathcal{P}
5	\mathcal{N}
7	\mathcal{N}
9	\mathcal{P}
11	\mathcal{P}
13	\mathcal{N}
15	\mathcal{P}
17	\mathcal{P}

In order to simplify the computation, we prove the following property :

Proposition 17. *If there is a winning strategy for the first player in 2-DISTANCE 2-COLORING on odd paths, the central vertex must be colored by the first player.*

Proof. Vertices are labeled from 1 to $2k+1$. Let s be the involution $s(i) = 2k+1 - i$. Suppose that the first player can win without coloring the middle vertex. For each vertex i colored by the first player, the second player can color $s(i)$ with the other color. Finally the second player will win the game. \square

Hence we cannot conclude the general case of 2-DISTANCE 2-COLORING on odd length paths, but we solve the other cases and the odd lengths from 1 to 17. This game is complex to study, particularly since it does not simplify into a sum of two smaller games.

Another indication for the complexity of this game is the computational hardness of the general version. We now show that the game is PSPACE-complete in general

Proposition 18. *Determining whether a position in 2-DISTANCE 2-COLORING is in \mathcal{N} is PSPACE-complete.*

Proof. Since all instances of the game are short and the number of plays is bounded above by the number of unpainted vertices, the problem is in PSPACE. As with the other hardness proofs in this paper, we will reduce from NODE-KAYLES. Again, the gadgets will restrict players to only have one color available, and also separate each original pair of adjacent vertices by an extra unpaintable vertex.

Given an instance of NODE-KAYLES, $G = (V, E)$, create a new graph, $G' = (V', E')$ where

$$V' = V \cup \{v_{e,0}, v_{e,1}, v_{e,2} : \forall e \in E\} \text{ and}$$

$$E' = \{(u, v_{e,0}), (w, v_{e,0}), (v_{e,0}, v_{e,2}), (v_{e,2}, v_{e,1}) : \forall (u, w) = e \in E\}.$$

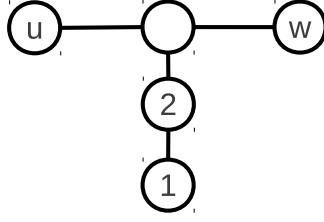


FIGURE 4. Reduction of a NODE-KAYLES edge (u, w) to 2-DISTANCE 2-COLORING. Numbers indicate colors for vertices.

To complete the reduction, we paint all vertices $v_{e,1}$ with color 1 and all vertices $v_{e,2}$ with color 2. In the resulting graph, *all* vertices are within distance 2 of a vertex painted with color 2, and all vertices not in V are within distance 2 of a vertex painted with color 1. Thus, only vertices in V can be painted (and only with color 1). These are all within distance 2 of another vertex from V exactly when those vertices were adjacent in G . Thus, each move in this game is equivalent to the corresponding move on the NODE-KAYLES instance, meaning this game is also PSPACE-hard.

Thus, determining whether a position of 2-DISTANCE 2-COLORING is in \mathcal{N} is PSPACE-complete. \square

We can expand on this by providing for any number of colors.

Corollary 19. 2-DISTANCE k -COLORING is PSPACE-complete for any number of colors, k .

Proof. For each additional color, add another node painted that color adjacent to $v_{e,0}$ for each edge $e \in E$. Now that color is not available to paint any vertices on the new graph. \square

Open Problems. Among the different games listed in this paper, this ruleset is certainly the hardest one to solve. The investigation of odd paths remains for us paramount to go further on it.

6. SEQUENTIAL COLORING (ALSO CALLED ONLINE COLORING)

A sequential coloring is based on assigning colors to vertices in some predetermined order. Games based on sequential colorings were introduced by Bodlaender [3] in 1989, where he presented very different results for the separate cases $k = 2$ and $k \geq 3$. To the best of our knowledge, the sequential coloring game is the only variant that has been previously considered in the literature on impartial coloring games.

Definition 7. SEQUENTIAL k -COLORING is an impartial coloring ruleset which includes a sequential function, $f : \{1, \dots, |V|\} \mapsto V$. On the i -th turn, the current player *must* properly paint the vertex $f(i)$. In other words, the only legal moves must paint the uncolored vertex with the lowest index, and this coloring must remain proper; no neighboring vertices may be painted the same color.

For $k \geq 3$, Bodlaender [3] finds the game to be computationally hard.

Proposition 20. *For $k \geq 3$, it is PSPACE-complete to determine whether an position of SEQUENTIAL k -COLORING is in \mathcal{N} [3].*

The remainder of this section considers the case where $k = 2$. In this case, Bodlaender presents a polynomial-time algorithm to solve the game on any graph. The complexity of his algorithm is about $O(n + e\alpha(e, n))$ where α is the inverse Ackermann function, and n the order of the graph.

According to this definition, player 1 colors the vertices $f(2i + 1)$, and player 2 the vertices $f(2i)$ for $i \geq 0$. Although Bodlaender's algorithm is very efficient, we here propose a better algorithm for paths and cycles, of complexity $O(n)$. Let P_n be the path of length n , with the vertices labeled from 1 to n , and consider SEQUENTIAL 2-COLORING on P_n , with a given sequential function f . We denote $g = f^{-1}$.

Definition 8. We classify the set of vertices of P_n into three types :

- Source vertices: these are local minima of f . v is a source vertex if and only if $g(v) < g(v_1)$ and $g(v) < g(v_2)$, where v_1 and v_2 are adjacent vertices of v . Note that if v is an extremity of P_n , v_2 does not exist and only the first condition is considered. Roughly speaking, source vertices are those that are colored before their neighbours.
- Closed vertices: these are local maxima of f . Let v_1 and v_2 be adjacent vertices of v . v is a closed vertex if and only if $g(v) > g(v_1)$ and $g(v) > g(v_2)$. Roughly speaking, if v_1 and v_2 have different colors, the player who colors a closed vertex v loses the game.
- Constrained vertices: these are neither maxima or minima of f . Let v_1 and v_2 be two adjacent vertices of v (without loss of generality, we suppose $g(v_2) > g(v_1)$). v is a constrained vertex if and only if $g(v_2) > f(v) > g(v_1)$.

Proposition 21. *A player has to make a color choice only on source vertices. He may lose only when playing on closed vertices.*

Proof. For any vertex v which is not a source, at least one adjacent vertex of v has been previously painted. On constrained vertices, there is exactly one vertex colored, which means the color of v is imposed. If v is a closed vertex, either v is not colorable (if the two adjacent vertices are holding different colors), or the color of v is imposed. \square

Theorem 22. *Given a path P of length n and a sequential function f , we can decide in time $O(n)$ whether this position is in \mathcal{P} or \mathcal{N} .*

Proof. According to Proposition 21, a player may lose only when he must play on a closed vertex, and he makes choices only on source vertices. This explains why we will consider only source and closed vertices, and we can remove the constrained ones from the path, by connecting their two adjacent vertices. For example, let us consider the game on a path P of length 11, using the sequential function f below (recall that $f^{-1} = g$): We are now looking for the player who will lose. Hence we have to consider closed vertices, ordered by increasing values of f . On the above example, the first closed vertex u_0 has the value 9 according to f . Since 9 is odd, the first player will color it. Consider its two adjacent vertices v_0 and v_1 , with

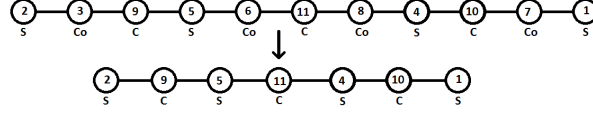


FIGURE 5. Removing constrained vertex. Labels on vertices are their value using f^{-1} , S= Source, C = Closed, Co = Constrained

respectively the values 2 and 5. As v_0 will be colored first, the first player will be able to color u_0 if and only if he owns v_1 . If he does not, the other player can color v_0 and v_1 in a such way the first player will lose on u_0 (by coloring v_0 and v_1 with the same color).

As $f(v_1) = 5$, the first player owns v_1 and he will be able to color u_0 by choosing the appropriate color on v_1 . Consequently, the first player does not have the choice when coloring v_1 and u_0 , and we can delete them as if they were constrained vertices:

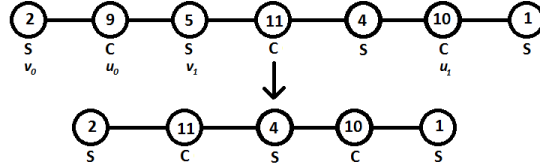


FIGURE 6. Removing vertices 9 and 5, since the first player has no choice.

By considering the next closed vertex (u_1 with the value 10), and with the same argument, as the second player owns the vertices 10(v_3) and 4(v_2), we also delete them.

Then, when considering the last vertices 11, 1 and 2 (respectively u_2, v_4, v_5 on the figure), one can see that the parity of v_5 is not the same as u_2 , hence the owner of the vertices is not the same. Therefore, u_2 can not be colored by the first player, who will lose the game.

The algorithm to decide who the winner is has five steps:

- Classify the vertices into source, constrained and closed.
- Remove the constrained vertices.
- Order the closed vertices by increasing values of the f function, and pick them up in this order.
- On each closed vertex u picked up, analyse the two adjacents source vertices. If the owner of the vertex v with the higher value of f is the same as u , then remove u and v and repeat this operation. Otherwise the game is losing for the owner of u .
- If there are no more vertices left, and no result yet, the game is winning for the first player if the length of the path is odd. Otherwise it is winning for the second player.

Each step take $O(n)$ operations, especially the ordering because we know the maximum value of f . \square

7. CONCLUSIONS

This work contains many new results concerning impartial coloring games as alternatives to the strictly partisan graph coloring games of COL and SNORT. We define six new rulesets concerning games that can use any number of colors, two of which, ORIENTED k -COLORING (definition 3) and ORIENTED BLUE-RED-COLORING (definition 4) are specific to directed graphs. As far as the authors are aware, only SEQUENTIAL k -COLORING has been studied prior to this work.

For many of the rulesets, we study the winnability of specific graph types. For paths, we show how to find the outcome class for starting positions of PROPER 2-COLORING and for even-length paths in 2-DISTANCE 2-COLORING.

For cycles, we solve the starting positions for PROPER 2-COLORING, ORIENTED BLUE-RED-COLORING, odd cycles for WEAK 2-COLORING and even cycles for 2-DISTANCE 2-COLORING. The result for WEAK 2-COLORING determines the outcome class for all of positions of odd cycles, not just starting positions. Furthermore, the game is trivial in this case: the Grundy values are all either 0 or 1, meaning that player strategies will not affect the outcome of the game.

For those graphs that have an involution, we show cases which can determine the outcome class of starting positions of PROPER k -COLORING.

Concerning the computational complexity of determining the outcome class, we show that four of these games are PSPACE-complete: PROPER k -COLORING, ORIENTED k -COLORING, ORIENTED BLUE-RED-COLORING and d -DISTANCE k -COLORING for the case where $d = 2$. For SEQUENTIAL 2-COLORING, we improve upon a previous result by finding a new algorithm to determine the outcome class of the game in $O(n)$ time. The complexities of d -DISTANCE k -COLORING for $d > 2$ and WEAK k -COLORING remain open.

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